

CORRECT OBSERVER'S EVENT HORIZON IN DE SITTER SPACE-TIME

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ABSTRACT. A correct description of the past and the future event horizons for timelike geodesics in de Sitter space-time of the first kind is given.

Keywords and phrases: de Sitter space-time of the first kind, event horizon, globally hyperbolic space-time, Lorentz manifold, world line.

1. INTRODUCTION AND MAIN RESULTS

In this paper we continue to study properties of the future and the past event horizons for any time-like geodesic in de Sitter space-time of the first kind $S(R)$ announced earlier in the paper [1]. Later we shall use shorter term "de Sitter space-time".

In Section 2 we give necessary definitions and known results about globally hyperbolic space-time.

In Section 3 are given exact description of the past event horizon for every time-like geodesic of de Sitter space-time. We prove the theorem announced earlier in the paper [1] as the theorem 8. This theorem sets a connection of the future and the past event horizons between themselves and with the so-called Lobachevsky space of positive curvature $\frac{1}{R^2}$ in the sense of B.A. Rosenfeld (see p. 155 in [4]), which is obtained by the gluing of antipodal events in $S(R)$. In this paper we give correct figures (see Fig. 2, 3) of an observer's event horizon, which differ with the corresponding non-correct figure (see. Fig. 1) from the Hawking's book [6] on the page 120.

2. PRELIMINARIES

We now remind necessary definitions from [3], [5].

Let M be a C^∞ -manifold of dimension $n + 1 \geq 2$. A *Lorentzian metric* g for M is a smooth symmetric tensor field of the type $(0, 2)$ on M which assigns to each point $p \in M$ a nondegenerate inner product $g|_p: T_p M \times T_p M \rightarrow \mathbb{R}$ of the signature $(+, +, \dots, +, -)$. Then the pair (M, g) is said to be *Lorentzian manifold*.

A nonzero tangent vector v is said to be respectively *time-like*, *space-like*, or *isotropic* if $g(v, v)$ is negative, positive, or zero. A tangent vector v is said to be *non-space-like* if it is time-like or isotropic. A continuous vector field X on Lorentzian manifold M is called *time-like* if $g(X(p), X(p)) < 0$ for all events $p \in M$. If Lorentzian manifold (M, g) admits a time-like vector field X , then we say that (M, g) is *time oriented by the field* X . The time-like vector field X separates all non-space-like vectors into two disjoint classes of *future directed* and *past directed* vectors. More exactly, a non-space-like vector $v \in T_p M$, $p \in M$, is said to be *future directed* (respectively, *past directed*) if

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$g(X(p), v) < 0$ (respectively, $g(X(p), v) > 0$). A Lorentzian manifold is *time orientable* if it admits some time-like vector field X .

Definition 1. A space-time (M, g) is a connected Hausdorff manifold of dimension equal or greater than two with a countable basis supplied with a Lorentzian C^∞ -metric g and some time orientation.

A continuous piecewise smooth curve (path) $c = c(t)$ with $t \in [a, b]$ or $t \in (a, b)$ on Lorentzian manifold (M, g) is said to be *non-space-like* if $g(c'_l(t), c'_r(t)) \leq 0$ for every inner point t from the domain of the curve c , where $c'_l(t)$ (respectively, $c'_r(t)$) denotes left (respectively, right) tangent vector. If (M, g) is a space-time, then the curve $c = c(t)$ with $t \in [a, b]$ or $t \in (a, b)$, is *future directed* or *past directed*, i.e. all (one-sided) tangent vectors of the curve c either are directed to the future or are directed to the past. The *causal future* $J^+(L)$ (respectively, the *causal past* $J^-(L)$) of a subset L of the space-time (M, g) is defined as the set of all events $q \in M$, for which there exists future directed (respectively past directed) curve $c = c(t)$, $t \in [a, b]$, such that $c(a) \in L$, $c(b) = q$. If $p \in M$, then we will use reduced notation $J^+(p)$ and $J^-(p)$ instead of $J^+(\{p\})$ and $J^-(\{p\})$.

Definition 2. [3] An open set U in a space-time is said to be *causally convex* if no non-space-like curve intersects U in a disconnected set. The space-time (M, g) is said to be *strongly causal* if each event in M has arbitrarily small causally convex neighborhoods.

In [3] it has been proved the following important proposition 2.7.

Proposition 1. A space-time (M, g) is strongly causal if and only if the sets of the form $I^+(p) \cap I^-(q)$ with arbitrary $p, q \in M$ form a basis of original topology (i.e. the Alexandrov topology induced on (M, g) agrees with given manifold topology).

Definition 3. A space-time (M, g) is called *globally hyperbolic* if it is strongly causal and satisfies the condition that $J^+(p) \cap J^-(q)$ is compact for all $p, q \in M$.

Definition 4. Let S be a subset in a globally hyperbolic space-time (M, g) . Then $\Gamma^-(S)$ (respectively, $\Gamma^+(S)$) denotes the boundary of the set $J^-(S)$ (respectively, $J^+(S)$) and is called the *past event horizon* (respectively, the *future event horizon*) of the set S .

A simplest example of a globally hyperbolic space-time is the *Minkowski space-time* $Mink^{n+1}$, $n+1 \geq 2$, i.e. a manifold \mathbb{R}^{n+1} , $n+1 \geq 2$, with the Lorentz metric g which has the constant components g_{ij} :

$$g_{ij} = 0, \text{ if } i \neq j; \quad g_{11} = \dots = g_{nn} = 1, \quad g_{(n+1)(n+1)} = -1.$$

in natural coordinates (x_1, \dots, x_n, t) on \mathbb{R}^{n+1} . The time orientation is defined by the vector field X with components $(0, \dots, 0, 1)$ relative to canonical coordinates in \mathbb{R}^{n+1} .

A more interesting example is *de Sitter space-time*. It is easily visualized as the hyperboloid $S(R)$ with one sheet

$$(1) \quad \sum_{k=1}^n x_k^2 - t^2 = R^2, \quad R > 0,$$

in Minkowski space-time $Mink^{n+1}$, $n+1 \geq 3$, with Lorentzian metric induced from $Mink^{n+1}$.

The *Lorentz group* is the group of all linear isometries of the space $Mink^{n+1}$, transforming to itself the «upper» sheet of the hyperboloid $\sum_{k=1}^n x_k^2 - t^2 = -1$ (which is

isometric to the Lobachevsky space of the constant sectional curvature -1). The Lorentz group acts transitively by isometries on $S(R)$.

The time orientation on $S(R)$ is defined by unit tangent vector field Y such that Y is orthogonal to all time-like sections

$$S(R, c) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : t = c\}, \quad c \in \mathbb{R}.$$

Notice that every integral curve of the vector field Y is a future directed time-like geodesic in $S(R)$. Therefore we can consider it as a world line of some observer.

3. MAIN RESULT

The main result of this paper is the following

Theorem 1. *Let L be a time-like geodesic in de Sitter space-time, $\Gamma^-(L)$ is the past event horizon for L (observer's event horizon). Then*

1. $\Gamma^-(L) = S(R) \cap \alpha$, where α is some hyperspace in \mathbb{R}^{n+1} which goes through the origin of coordinate system, consists of isotropic geodesics.
2. $J^+(L) = -J^-(-L)$, $J^-(L) = -J^+(-L)$.
3. The sets $J^-(L)$ and $J^+(-L)$ (respectively $J^+(L)$ and $J^-(-L)$) don't intersect and have joint boundary $\Gamma^-(L)$. In particular, the past event horizon for L coincides with the future event horizon for $-L$ (respectively, the future event horizon for L coincides with the past event horizon for $-L$).
4. The quotient map $pr : S(R) \rightarrow S_n^1(R)$, gluing antipodal events in $S(R)$, is diffeomorphism on all open submanifolds $J^+(L)$, $J^-(-L)$, $J^-(L)$, $J^+(-L)$ which identifies antipodal events of boundaries for these submanifolds (i.e. the corresponding event horizons).
5. The quotient manifold $(S_n^1(R), G)$, where $g = pr^*G$, is the Lobachevsky space of positive curvature $\frac{1}{R^2}$ in the sense of B.A.Rosenfeld (see p. 155 in [4]).

Proof. Notice that if L and L' are time-like geodesics in $S(R)$ and $p \in L$, $p' \in L'$, then there exists preserving the time direction isometry i of de Sitter space-time such that $i(p) = p'$ and $i(L) = L'$. Therefore i translates the past event horizon of L to the past event horizon of L' . Therefore it is enough to prove points 1–4 of theorem 1 only for one time-like geodesic.

We will suppose that L is integral curve of the vector field Y , which intersects $S(R, 0)$ at the event p with Descartes coordinates $(R, 0, \dots, 0)$. Let us show that

$$(2) \quad \Gamma^-(L) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : x_1 = t\}.$$

Denote by C_p isotropic cone at the point p with Descartes coordinates $(R, 0, \dots, 0)$. It follows from (1) that

$$(3) \quad C_p = \{(x_1, \dots, x_n, t) \in S(R) \mid x_1 = R\}.$$

Note that the causal past $J^-(p)$ of the event p is the region in $S(R)$, lying in the hyperspace $t < 0$ and bounding by isotropic cone C_p .

For any $\psi \in \mathbb{R}$, the restriction of the Lorentz transformation $\Phi_\psi : S(R) \rightarrow S(R)$, realizing hyperbolic rotation in two-dimensional plane Ox_1t , prescribed by formulae

$$(4) \quad x'_1 = x_1 \operatorname{ch} \psi + t \operatorname{sh} \psi; \quad x'_i = x_i, \quad i = 2, \dots, n; \quad t' = x_1 \operatorname{sh} \psi + t \operatorname{ch} \psi,$$

is an isometry of the space-time $S(R)$, preserving the time direction. The set Φ of all such transformations Φ_ψ , $\psi \in \mathbb{R}$, forms a one-parameter subgroup of the Lorentz

group. Here the orbit of the event p with respect to Φ coincides (up to parametrization) with the curve L , and the event $L(\psi) := \Phi_\psi(p)$ has Descartes coordinates $(R \operatorname{ch} \psi, 0, \dots, 0, R \operatorname{sh} \psi)$.

Under the action of Φ_ψ , isotropic cone C_p at the point p passes to isotropic cone $C_{L(\psi)}$ at the point $L(\psi)$ and the causal past $J^-(p)$ of the event p passes to the causal past $J^-(L(\psi))$ of the event $L(\psi)$. It follows from (3), (4) that

$$(5) \quad C_{L(\psi)} = \left\{ (x_1, \dots, x_n, t) \in S(R) \mid x_1 - t \operatorname{th} \psi = \frac{R}{\operatorname{ch} \psi} \right\}.$$

Note that the set $J^-(L)$ of observed events for L is the union of all sets $J^-(L(\psi))$, where $\psi \in \mathbb{R}$. If $\psi_1 < \psi_2$ then $L(\psi_1) \in J^-(L(\psi_2))$, therefore $J^-(L(\psi_1)) \subset J^-(L(\psi_2))$. Hence the past event horizon of L is limiting position of the «lower» half (lying in the hyperspace $t < \frac{x_1 \operatorname{sh} \psi}{\operatorname{ch} \psi}$) of isotropic cone $C_{L(\psi)}$ when $\psi \rightarrow +\infty$. Now (2) follows from (5) and the fact that $\operatorname{ch} \psi \rightarrow +\infty$, $\operatorname{th} \psi \rightarrow 1$ when $\psi \rightarrow +\infty$.

Note also that the future event horizon $\Gamma^+(L)$ for L is limiting position of «upper» half (lying in the hyperspace $t > \frac{x_1 \operatorname{sh} \psi}{\operatorname{ch} \psi}$) of isotropic cone $C_{L(\psi)}$ when $\psi \rightarrow -\infty$. Then by (5),

$$(6) \quad \Gamma^+(L) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : x_1 + t = 0\}.$$

It follows from (2), (6) that

$$(7) \quad J^-(L) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : x_1 - t > 0\},$$

$$(8) \quad J^+(L) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : x_1 + t > 0\}.$$

To prove the point 2 of theorem 1, it is enough to note that central symmetry i_0 of the space-time \mathbb{R}^{n+1} relative to the origin of coordinate system is an isometry of de Sitter space-time, reversing the time direction. Therefore $i_0(J^-(p)) = J^+(-p)$, $i_0(J^+(p)) = J^-(-p)$ for any event $p \in L$. Corresponding equalities in the point 2 of theorem 1 follow from here.

On the ground of p. 2 in theorem 1 and (7), (8),

$$(9) \quad J^+(-L) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : x_1 - t < 0\},$$

$$(10) \quad J^-(-L) = S(R) \cap \{(x_1, \dots, x_n, t) \in \mathbb{R}^{n+1} : x_1 + t < 0\}.$$

The statements of p. 3 in theorem 1 are valid in view of (2), (6) – (10).

Let us prove p. 4 of theorem 1. It follows from (7), (8) that every set $J^-(L)$, $J^+(L)$ is open and contains no pair of antipodal events. Conversely, in consequence of (2), (6), the past event horizon $\Gamma^-(L)$ and the future event horizon $\Gamma^+(L)$ for L are centrally symmetric sets. Therefore the quotient map $pr : S(R) \rightarrow S_n^1(R)$, identifying antipodal events from $S(R)$, is a diffeomorphism on the open submanifold $J^-(L)$ ($J^+(L)$) and glues antipodal events of the set $\Gamma^-(L)$ ($\Gamma^+(L)$). Now the rest statements of p. 4 in theorem 1 follow from equalities in p. 2 of this theorem.

P. 5 of theorem 1 is an immediate corollary of statements in p. 4. \square

Corollary 1. *Let L be a time-like geodesic in $S(R)$, p is the joint event of L and $S(R, 0)$. Then the past event horizon $\Gamma^-(L)$ for L intersects $S(R, 0)$ by the sphere $S_{S(R, 0)}(p, \pi R/2)$ of the radius $\pi R/2$ with the center p .*

Proof. Using the argument in the proof of theorem 1, we can assume without loss of generality that L is integral curve of vector field Y , intersecting $S(R, 0)$ at the point p

with Descartes coordinates $(R, 0, \dots, 0)$. Now the corollary 1 follows from (2) and the fact that

$$S_{S(R,0)}(p, \pi R/2) = \{(x_1, \dots, x_n, t) \in S(R) \mid x_1 = 0, t = 0\}.$$

□

One can consider the time-like geodesic L above as the history (or the world line) of an *eternal* observer.

The Fig. 4.18, p. 120 in the Hawking's book [6] (Fig. 1 in our paper) admits two interpretations, namely as a picture of the history and corresponding (past) event horizon for a *real* or an *eternal* observer in de Sitter space-time.

The first interpretation corresponds to the inscription "Surface of constant time" but then contradicts to the smoothness of bright region at its top point since the top point must be the cone point for this region. To avoid the last mistake on Fig. 4.18, it would better to take the second interpretation and change the inscription "Surface of constant time" by the inscription " $t = \infty$ ", assuming that the scale on the picture goes to zero when $t \rightarrow \infty$. But for the second interpretation, the observer's event horizon is depicted incorrectly since on the ground of the corollary 1, it must intersect the «throat» of the hyperboloid of one sheet (the sphere $S(R, 0)$) by the sphere $S_{S(R,0)}(p, \pi R/2)$, where p is the intersection event of the world line L with $S(R, 0)$; for the first interpretation, the above intersection must be $S_{S(R,0)}(p, r)$, where $r < \pi R/2$. On the other hand, this intersection on Fig. 4.18 is empty.

The correct picture for the second interpretation for bounded (respectively, infinite) time is given on our Fig. 2 (respectively, Fig. 3). Note also that the observer's event horizon (see Fig. 2) consists of all isotropic geodesics, lying in corresponding hyperspace in $Mink^{n+1}$ going through zero event.

Earlier V.N.Berestovskii formulated without proof the above statements about past event horizon in his plenary talk, but in the text [2] of this talk they are absent.

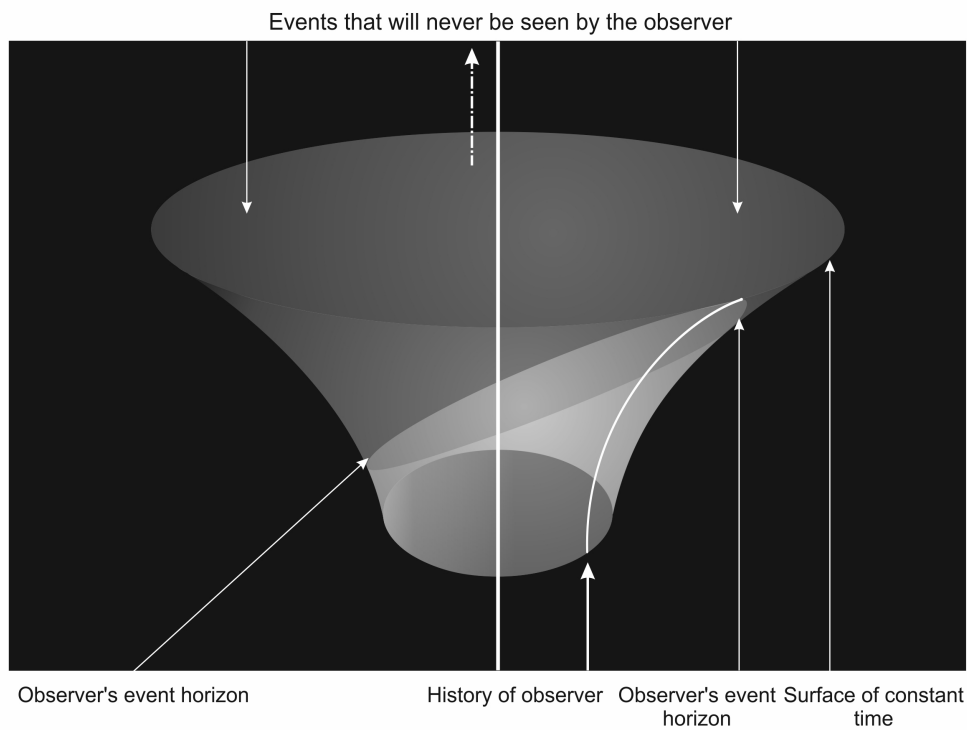
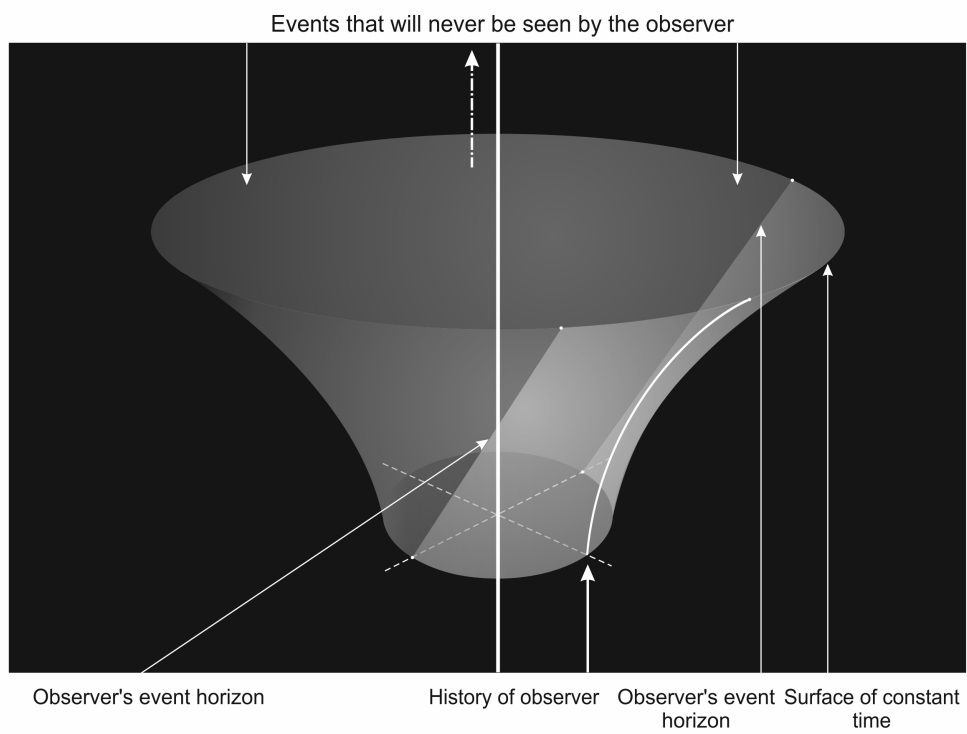


FIGURE 1. Fig. 4.18 of (real) observer's event horizon from Hawking's book [6]

FIGURE 2. Correct (eternal) observer's event horizon for $0 \leq t \leq t_0$

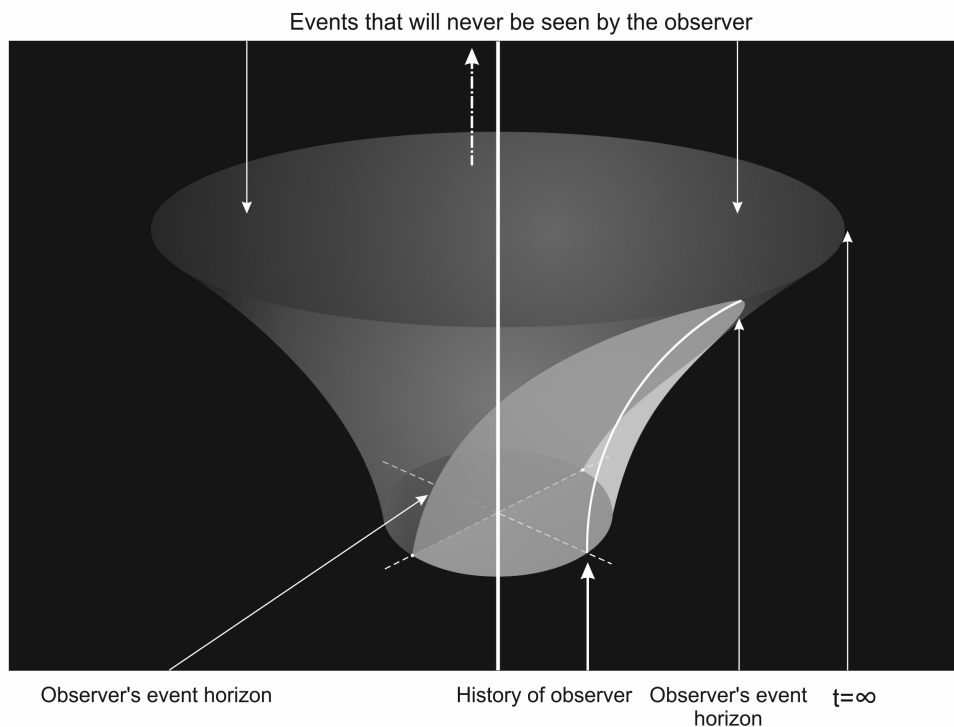


FIGURE 3. (Eternal) observer's event horizon for $0 \leq t \leq \infty$

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